# Lecture 11

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Ex: $Q = -x^2 + y^2 + 4yz + 4zx$. Reduce it to canonical form and find the values of $x$ for which Q = 0.

$$Q = X^T AX = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 0 & 2 \\ 0 & 1-\lambda & 2 \\ 2 & 2 & -\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)[(-\lambda)(1-\lambda) - 4] + 2[(-2)(1-\lambda)]$$

$$\Rightarrow \lambda^3 - 9\lambda = 0$$

$$\Rightarrow \lambda = 0, 3, -3$$

For $\lambda = 0$,

$$(A - 0 \cdot I)X = 0$$

$$\begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 0 \end{pmatrix} X = 0$$

$R_3 \longrightarrow R_3 + 2R_1$
$R_3 \longrightarrow R_3 - 2R_2$

$$\begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

if $x = k$, then $-x + 2z = 0$ & $y + 2z = 0$

$$\Rightarrow X_1 = \begin{bmatrix} k & -k & k/2 \end{bmatrix}^T$$

take k = 2,

$$X_1 = \begin{bmatrix} 2 & -2 & 1 \end{bmatrix}^T$$

For $\lambda = 3$,

$$(A - 3 \cdot I)X = 0$$

$$\begin{pmatrix} -4 & 0 & 2 \\ 0 & -2 & 2 \\ 2 & 2 & -3 \end{pmatrix} X = 0$$

$R_3 \longrightarrow 2R_3 + R_1$
$R_3 \longrightarrow R_3 + 2R_2$

$$\begin{pmatrix} -4 & 0 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

if $x = k$, then $-4x + 2z = 0$ & $-2y + 2z = 0$

$$\Rightarrow X_2 = [k \quad 2k \quad 2k]^T$$

take k = 1,

$$X_2 = [1 \quad 2 \quad 2]^T$$

For $\lambda = -3$,

$$(A + 3 \cdot I)X = 0$$

$$\begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 2 \\ 2 & 2 & 3 \end{pmatrix} X = 0$$

$R_3 \longrightarrow R_3 - R_1$
$R_3 \longrightarrow 2R_3 - R_2$

$$\begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

if $x = k$, then $2x + 2z = 0$ & $2y + z = 0$

$$\Rightarrow X_1 = [k \quad k/2 \quad -k]^T$$

take k = 2,

$$X_1 = [2 \quad 1 \quad -2]^T$$

Thus,

$$\Rightarrow \frac{X_1}{||X_1||} = \frac{1}{3} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \frac{X_2}{||X_2||} = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \frac{X_3}{||X_3||} = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$P = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \frac{X_1}{||X_1||} & \frac{X_2}{||X_2||} & \frac{X_3}{||X_3||} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

$$Q = X^T AX = Y^T(P^T AP)Y = Y^{-1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix} Y$$

Thus the canonical form $Q = 3y_2^2 - 3y_3^2$.
⇒ Indefinite quadratic form.

Now lets calculate the value of X for which Q = 0.

$$Q = 3y_2^2 - 3y_3^2 = 0$$

$$\Rightarrow y_2 = +_- y_3$$

$$Y = [y_1 \quad y_2 \quad y_3]^T = [k_1 \quad k_2 \quad +k_2]^T, k_1, k_2 \in \mathbb{R}$$

$$X = PY = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{-2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{-2}{3} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

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## Index

The number of positive square terms in the canonical form is called the index of Q, denotes by (s).

## Signature

The difference of number of positive and negative square tems is called the signature of the quadratic form denoted by $sig$.

$$sigQ = s - (r - s) = 2s - r$$

where,
r $\longrightarrow$ rank of matrix A.
s $\longrightarrow$ number of positive square terms.
(r - s) $\longrightarrow$ number of negative terms in canonical form.

The quadratic for is said to be

1. Positive definite if r = n, s = n
2. Negative definite if r = n, s = 0
3. Positive semi definite if r < n, s = r

4. Negative semi definite if $r < n$, $s = 0$
5. Indefinite in every other case.

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#semester-1 #mathematics #matrices