Lecture 10

How to classify Qaudratic form?

First we need to transform quadratic form into matrix form $Q = X^TAX$. Then we can transform

$$Q = \sum_{i}^{n}\sum_{j}^{n} a_{ij}x_ix_j \qquad \text{into} \qquad \sum_{i=1}^{n} b_iy_i^2$$

which will have only the square terms called the *canonical form*. This way it would be easy to classify the quadratic form just by looking at the coefficients of the term.

Taking X = PY,

$$Q = X^TAX = (PY)^TA(PY) = Y^TP^TAP$$
$$= Y^T(P^TAP)Y$$
$$= Y^TDY$$

we need to choose P such that D is a diagonal matrix. Thus when A is a symmetric matrix, we can diagonalize the matrix A to get $D = P^TAP$.

$$\Rightarrow \qquad P = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \frac{X_1}{||X_1||} & \frac{X_2}{||X_2||} & \cdots & \frac{X_n}{||X_n||} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}, \qquad D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

If all $\lambda_i > 0$ then $Q = Y^TDY = \sum \lambda_i y_i^2 > 0, Q = 0$ iff Y = 0
⇒ *Positive definite form*.

If any of the $\lambda_i = 0$ & $\lambda_i > 0$ for all other i,
⇒ *Positive semi definite form*.

If all $\lambda_i < 0$ then $Q = Y^TDY = \sum \lambda_i y_i^2 < 0, Q = 0$ iff Y = 0
⇒ *Negative definite form*.

If $\lambda \leq 0$ and atleast one $\lambda_i = 0$
⇒ *Negative semi definite form*.

Ex: for n = 2 & k > 0,

$$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 = k \quad \forall \quad \lambda_i > 0$$

we get the equation of ellipse in 2-D plane.

Ex: Reduce the quadratic form $6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4xz$ into canonical form by an orthogonal transformation.

$$Q = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = X^T A X$$

where,

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \& \quad A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

we get $\lambda = 2, 2, 8$.

Finding eigen vector for $\lambda = 8$ gives $X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \frac{X_1}{||X_1||} = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$.

for $\lambda = 2$, $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \frac{X_2}{||X_2||} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

for another $\lambda = 2$, $X_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \Rightarrow \frac{X_3}{||X_3||} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{-2}{\sqrt{5}} \end{bmatrix}$

$$\Rightarrow P = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-2}{\sqrt{5}} \end{bmatrix}$$

Since P is orthogonal, $P^{-1} = P^T$.

$$\Rightarrow P^{-1}AP = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$Q = Y^TAY = Y^T \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} Y$$

$$= 8y_1^2 + 2y_2^2 + 2y_3^2$$

Thus Q > 0 $\forall \quad Y \neq 0$ and Q = 0 iff Y = 0 is positive definite form.

#semester-1 #mathematics #matrices